

# True Global Optimality of the Pressure Vessel Design Problem: A Benchmark for Bio-Inspired Optimisation Algorithms

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## Abstract

The pressure vessel design problem is a well-known design benchmark for validating bio-inspired optimization algorithms. However, its global optimality is not clear and there has been no mathematical proof put forward. In this paper, a detailed mathematical analysis of this problem is provided that proves that 6059.714335048436 is the global minimum. The Lagrange multiplier method is also used as an alternative proof and this method is extended to find the global optimum of a cantilever beam design problem.

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## 1 Introduction

Engineering optimization is often non-linear with complex constraints, which can be very challenging to solve. Sometimes, seemingly simple design problems may in fact be very difficult indeed. Even in very simple cases, analytical solutions are usually not available, and researchers have struggled to find the best possible solutions. For example, the well-known design benchmark of a pressure vessel has only four design variables [[Cagnina et al. (2008), Gandomi et al. (2013), Yang (2010)]]; however, the global optimum solution for pressure vessel design benchmark is still unknown to the research community, despite a large number of attempts and studies, i.e. [Annaratone (2007), Deb and Gene (1997)]. Thus, a mathematical analysis will help to gain some insight into the problem and thus guide researchers to validate if their solutions are globally optimal. This paper attempts to provide a detailed mathematical analysis of the pressure vessel problem and find its global optimum. To the best of the author's knowledge, this is a novel result in the literature.

Therefore, the rest of the paper is organised as follows: Section 2, introduces the basic formulation of the pressure vessel design benchmark and then highlights the relevant numerical results from the literature. Section 3 provides an analysis of the global optimum for the problem, whereas Section 4 uses Lagrange multipliers as an alternative method to prove that the analysis in Section 3 indeed gives the global optimum. Section 5 extends the same methodology to analyse the optimal solution of another design benchmark: cantilever beam. Finally, we draw our conclusions in Section 6.

## 2 Pressure Vessel Design Benchmark

Bio-inspired optimization algorithms have become popular, and many new algorithms have emerged in recent years [[Yang and Gandomi (2012), Che and Cui (2011), Cui et al. (2013), Gandomi et al. (2012), Yang and Deb (2013), Yang (2013)]]. In order to validate new algorithms, a diverse set of test functions and benchmarks are often used [Gandomi and Yang (2011), Jamil and Yang (2013)]. Among the structural design benchmarks, the pressure vessel design problem is one of the most widely used.

In fact, the pressure vessel design problem is a well-known benchmark for validating optimization algorithms [[Cagnina et al. (2008), Yang (2010)]]. It has four design variables: thickness ( $d_1$ ), thickness of the heads ( $d_2$ ), the inner radius ( $r$ ) and the length ( $L$ ) of the cylindrical section. The main objective is to minimize the overall cost, under the nonlinear constraints of stresses and yield criteria. The thickness can only take integer multiples of 0.0625 inches.

This optimization problem can be written as

$$\begin{aligned} \text{minimize } f(\mathbf{x}) = & 0.6224d_1rL + 1.7781d_2r^2 \\ & + 3.1661d_1^2L + 19.84d_1^2r, \end{aligned} \tag{1}$$

subject to

$$\begin{cases} g_1(\mathbf{x}) = -d_1 + 0.0193r \leq 0 \\ g_2(\mathbf{x}) = -d_2 + 0.00954r \leq 0 \\ g_3(\mathbf{x}) = -\pi r^2L - \frac{4\pi}{3}r^3 + 1296000 \leq 0 \\ g_4(\mathbf{x}) = L - 240 \leq 0. \end{cases} \tag{2}$$

The simple bounds are

$$0.0625 \leq d_1, d_2 \leq 99 \times 0.0625, \quad 10.0 \leq r, L \leq 200. \tag{3}$$

Table 1: Summary of main results. [Results marked with \* are not valid, see text].

Authors	Results	Authors	Results
[Coello (2000a)]	6288.745	[Lee and Geem (2005)]	7198.433
[Li and Chou (1994)]	7127.3	[Cai and Thierauf (1997)]	7006.931
[Cagnina et al. (2008)]	<b>6059.714</b>	[Li and Chang (1998)]	7127.3
[Cao and Wu (1999)]	7108.616	[Hu et al. (2003)]	6059.131*
[He et al. (2004)]	<b>6059.714</b>	[Coello and Mezura Montes (2001)]	6059.946
[Huang et al. (2007)]	6059.734	[He and Wang (2006)]	6061.078
[Litinetskiand Abramzon (1998)]	7197.7	[Coello (2000b)]	6263.793
[Sandgren (1998)]	7980.894	[Kannan and Kramer (1994)]	7198.042
[Akhtar et al. (2002)]	6171	Yun [Yun (2005)]	7198.424
[Tsai et al. (2002)]	7079.037	[Cao and Wu (1997)]	7108.616
[Deb and Gene (1997)]	6410.381	[Coello (1999)]	6228.744
[Montes et al. (2007)]	6059.702*	[Parsopoulos and Vrahatis (2005)]	6544.27
[Shih and Lai (1995)]	7462.1	[Kaveh and Talatahari (2010)]	6059.73
[Sandgren (1990)]	8129.104	[Santos Ceolho (2010)]	<b>6059.714</b>
[Wu and Chow (1995)]	7207.494	[Rat and Liew (2003)]	6171
[Zhang and Wang (1993)]	7197.7	[Coello and Cortés (2004)]	6061.123
[Joines and Houck (1994)]	6273.28	[Michalewicz and Attia (1994)]	6572.62
[Hadj-Alouane and Bean (1997)]	6303.5	[Fu et al. (1991)]	8048.6
[Yang and Gandomi (2012)]	<b>6059.714</b>	[Gandomi et al. (2013)]	<b>6059.714</b>

This is a mixed-integer problem, which is usually challenging to solve. However, there are extensive studies in the literature, and details can be found in several good survey papers [[Thanedar and Vanderplaats (1995), Gandomi and Yang (2011)]]]. The main results are summarized in Table 1. It is worth pointing out that some results are not valid and marked with \* in the footnote. These seemingly lower results actually violated some constraints and/or used different limits.

As it can be seen from this table, the results vary significantly from the highest value of 8129.104 by Sandgren [Sandgren (1998)] to the lowest value of 6059.714 by a few researchers [[Cagnina et al. (2008), Santos Ceolho (2010), He et al. (2004), Gandomi et al. (2013), Gandomi et al. (2011), Yang and Gandomi (2012)]]]. However, nobody is sure that 6059.714 is the globally optimal solution for this problem.

The best solution by [Gandomi et al. (2013)] and [Yang and Gandomi (2012)] is

$$f_* = 6059.714, \quad (4)$$

with

$$\mathbf{x}_* = (0.8125, 0.4375, 42.0984, 176.6366). \quad (5)$$

The rest of the paper analyses this problem mathematically and proves that this solution is indeed near the global optimum and concludes that the true globally minimal solution is  $f_{\min} = 6059.714335048436$  at

$$\mathbf{x}_* = (0.8125, 0.4375, 42.0984455958549, 176.6365958424394). \quad (6)$$

### 3 Analysis of Global Optimality

As all the design variables must have positive values and  $f$  is monotonic in all variables, the minimization of  $f$  requires the minimization of all the variables if there is no constraint. As there are 4 constraints, some of the constraints may become tight or equalities. As the range of  $L$  is  $10 \leq L \leq 200$ , the constraint  $g_4$  automatically satisfies  $L \leq 240$  and thus becomes redundant, which means that the upper bound for  $L$  is

$$L \leq 200. \quad (7)$$

This is a mixed integer programming problem, which often requires special techniques to deal with the integer constraints. However, as the number of combinations of  $d_1$  and  $d_2$  is not huge (just  $100^2 = 10,000$ ), it is possible to go through all the cases for  $d_1$  and  $d_2$ , and then focus on solving the optimization problems in terms of  $r$  and  $L$ .

The first two constraints are about stresses. In order to satisfy these conditions, the hoop stresses  $d_1/r$  and  $d_2/r$  should be as small as possible. This means that  $r$  should be reasonably large. For any given  $d_1$  and  $d_2$ , the first two constraints become

$$r \leq \frac{d_1}{0.0193}, \quad r \leq \frac{d_2}{0.00954}. \quad (8)$$

So the upper bound or limit for  $r$  becomes

$$U_r = \min\left\{\frac{d_1}{0.0193}, \frac{d_2}{0.00954}\right\}. \quad (9)$$

The above argument that  $r$  should be moderately high, may imply that one of the first two constraints can become tight, or an equality.

The third constraint  $g_3$  can be rewritten

$$\pi r^2 L + \frac{4\pi}{3} r^3 \geq K, \quad K = 1296000. \quad (10)$$

In fact, this is essentially the requirement that the volume of the pressure vessel must be greater than a fixed volume. This provides the lower boundary in the search domain of  $(r, L)$ .

Since  $f(r, L)$  is monotonic in  $r$  and  $L$ , the global solution must be on the lower boundary for any given  $d_1$  and  $d_2$ . In other words, the inequality  $g_3$  becomes an equality

$$\pi r^2 L + \frac{4\pi}{3} r^3 = K. \quad (11)$$

Using equation (10), with  $L \leq 200$ ,  $r$  can be derived using Newton's method (or an online polynomial root calculator). The only positive root implies that

$$r \geq 40.31961872409872 = r_1. \quad (12)$$

Similarly,  $L \geq 10$  means that

$$r \leq 65.22523261350128 = r_2. \quad (13)$$

So the true value of  $r$  must lie in the interval of  $[r_1, r_2]$ .

From the first inequality with  $r = r_1$ , we have

$$d_1 \geq 0.7782. \quad (14)$$

The second inequality gives

$$d_2 \geq 0.3846. \quad (15)$$

As both  $d_1$  and  $d_2$  must be integer multiples  $I$  and  $J$ , respectively, of  $d = 0.0625$ , the above two inequalities mean

$$I = \lceil \frac{0.7782}{0.0625} \rceil = 13, \quad J = \lceil \frac{0.3846}{0.0625} \rceil = 7. \quad (16)$$

In other words, we have

$$d_1 \geq 13d = 0.8125, \quad d_2 \geq 7d = 0.4375. \quad (17)$$

From the objective function (Eq. 1), both  $d_1$  and  $d_2$  should be as small as possible, so as to get the minimum possible  $f$ . This means that the global minimum will occur at  $d_1 = 0.8125$  and  $d_2 = 0.4375$ .

Now the objective function with these  $d_1$  and  $d_2$  values can be written as

$$\begin{aligned} f(r, L) &= 0.5057rL + 0.77791875r^2 \\ &\quad + 2.090120703125L + 13.0975r. \end{aligned} \quad (18)$$

As  $d_1 = 0.8125$  and  $d_2 = 0.4375$ , the first inequalities ( $g_1$  and  $g_2$ ) will give an upper bound of  $r$

$$\begin{aligned} R_* &= \min\left\{\frac{d_1}{0.0193}, \frac{d_2}{0.00954}\right\} \\ &= \min\{42.0984455958549, 45.859538784067\} \\ &= 42.0984455958549. \end{aligned} \quad (19)$$

Again from the objective function, which is monotonic in terms of  $r$  and  $L$ , the optimal solution should occur at the two extreme ends of the boundary governed by Eq. (11).

The one end at  $r = R_*$  gives

$$L_* = \frac{K}{\pi R_*^2} - \frac{4R_*}{3} = 176.6365958424394. \quad (20)$$

This is the point for the global optimum with

$$f_{\min} = 6059.714335048436. \quad (21)$$

The other extreme point is at  $r' = 40.31961872409872$  and  $L = 200$ , which leads to an objective value of

$$f' = 6288.67704565344, \quad (22)$$

and clearly is not the global optimum.

## 4 Method of Lagrange Multipliers

The optimal solution (21) can alternatively be proved by solving the following constrained problem with one equality because all of the upper bounds or inequalities are automatical satisfied. By minimizing  $d_1$ , and  $d_2$ , the objective function becomes:

$$\begin{aligned} \text{minimize } f(r, L) &= 0.5057rL + 0.77791875r^2 \\ &+ 2.090120703125L + 13.0975r. \end{aligned} \quad (23)$$

subject to

$$g_e = \pi r^2 L + \frac{4\pi}{3}r^3 - K = 0, \quad (24)$$

with the simple bounds

$$\begin{aligned} 40.31961872409872 &\leq r \leq 42.098445595854919, \\ 10 &\leq L \leq 176.6365958424394. \end{aligned} \quad (25)$$

To avoid writing long numbers, let us define

$$\begin{aligned} a &= 0.5057, \quad b = 0.77791875, \\ c &= 2.090120703125, \quad d = 13.0975. \end{aligned} \quad (26)$$

We have

$$\text{minimize } f(r, L) = arL + br^2 + cL + dr. \quad (27)$$

This problem can be solved by the Lagrange multiplier method, and we have

$$\text{minimize } \phi = f + \lambda g_e, \quad (28)$$

where  $\lambda$  is the Lagrange multiplier.

The optimum should occur when

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \frac{\partial f}{\partial r} + \lambda \frac{\partial g_e}{\partial r} \\ &= aL + 2br + d + \lambda(2\pi rL + 4\pi r^2) = 0, \end{aligned} \quad (29)$$

$$\frac{\partial \phi}{\partial L} = \frac{\partial f}{\partial L} + \lambda \frac{\partial g_e}{\partial L} = ar + c + \lambda(\pi r^2) = 0, \quad (30)$$

$$\frac{\partial \phi}{\partial \lambda} = \pi r^2 L + \frac{4\pi}{3}r^3 - K = 0. \quad (31)$$

Now we have three equations for three unknowns

$$\begin{cases} aL + 2br + d + \lambda(2\pi rL + 4\pi r^2) = 0, \\ ar + c + \lambda(\pi r^2) = 0, \\ \pi r^2 L + \frac{4\pi r^3}{3} - K = 0. \end{cases} \quad (32)$$

The equation in the middle gives

$$\lambda = -\frac{ar + c}{\pi r^2}, \quad (33)$$

Substituting this, together with  $L = K/(2r^2) - 4r/3$ , into the first equation, we have

$$\begin{aligned} a(Kr - \frac{4\pi r^3}{3}) + 2b\pi r^4 + \pi dr^3 \\ - (ar + c)(2K + \frac{4\pi r^3}{3}) = 0, \end{aligned} \quad (34)$$

which is a quartic equation with four roots in general. The only feasible solution within  $[r_1, r_2]$  is 42.098445595854919, which corresponds to  $L = 176.6365958424394$ . This solution is indeed the global best solution as given in (21).

## 5 Cantilever Beam Design Benchmark

Another widely used benchmark for validating bio-inspired algorithms is the design optimization of a cantilever beam, which is to minimize the overall weight of a cantilever beam with square cross sections [[Fleury and Braibant (1986), Gandomi et al. (2013), Gandomi et al. (2011)]]]. It can be formulated as

$$\text{minimize } f(\mathbf{x}) = 0.0624(x_1 + x_2 + x_3 + x_4 + x_5), \quad (35)$$

subject to the inequality

$$g(\mathbf{x}) = \frac{61}{x_1^3} + \frac{37}{x_2^3} + \frac{19}{x_3^3} + \frac{7}{x_4^3} + \frac{1}{x_5^3} - 1 \leq 0. \quad (36)$$

The simple bounds/limits for the five design variables are

$$0.01 \leq x_i \leq 100, \quad i = 1, 2, \dots, 5. \quad (37)$$

Since the objective  $f(\mathbf{x})$  is linear in terms of all design variables, and  $g(\mathbf{x})$  encloses a hypervolume, it can be thus expected that the global optimum occurs when the inequality becomes tight. That is, the inequality becomes an equality

$$g(\mathbf{x}) = \frac{61}{x_1^3} + \frac{37}{x_2^3} + \frac{19}{x_3^3} + \frac{7}{x_4^3} + \frac{1}{x_5^3} - 1 = 0. \quad (38)$$

For ease of analysis, we rewrite the above equation as

$$g(\mathbf{x}) = \sum_{i=1}^5 \frac{a_i}{x_i^3} - 1 = 0, \quad (39)$$

where

$$\mathbf{a} = (a_1, a_2, a_3, a_4, a_5) = (61, 37, 19, 7, 1). \quad (40)$$

Hence, the cantilever beam problem becomes

$$\text{minimize } f(\mathbf{x}) = k \sum_{i=1}^5 x_i, \quad k = 0.0624, \quad (41)$$

subject to

$$g(x) = \sum_{i=1}^5 \frac{a_i}{x_i^3} - 1 = 0. \quad (42)$$

Using the method of Lagrange multipliers, we have

$$\text{Minimize } \phi = f + \lambda g$$

$$= k \sum_{i=1}^5 x_i + \lambda \left( \sum_{i=1}^5 \frac{a_i}{x_i^3} - 1 \right). \quad (43)$$

Then, the optimality conditions give

$$\frac{\partial \phi}{\partial x_i} = k + \lambda(-3) \frac{a_i}{x_i^4} = 0, \quad i = 1, 2, \dots, 5, \quad (44)$$

$$\frac{\partial \phi}{\partial \lambda} = \sum_{i=1}^5 \frac{a_i}{x_i^3} - 1 = 0. \quad (45)$$

From Eq. (44), we have

$$x_i^4 = \frac{3\lambda a_i}{k}, \quad \text{or} \quad \frac{a_i}{x_i^3} = \frac{kx_i}{3\lambda}. \quad (46)$$

Substituting it into Eq. (45), we have

$$\sum_{i=1}^5 \left( \frac{kx_i}{3\lambda} \right) - 1 = \frac{k}{3\lambda} \left( \sum_{i=1}^5 x_i \right) - 1 = 0. \quad (47)$$

After rearranging and using results from (46), now we have

$$\frac{3\lambda}{k} = \sum_{i=1}^5 \left( \frac{3\lambda a_i}{k} \right)^{1/4}, \quad (48)$$

which is a nonlinear equation for  $\lambda$ . However, it is straightforward to find that

$$\lambda \approx 0.4466521202, \quad (49)$$

which leads to the optimal solution

$$\begin{aligned} \mathbf{x}_* = & (6.0160159, 5.3091739, \\ & 4.4943296, 3.5014750, 2.15266533), \end{aligned} \quad (50)$$

with the minimum

$$f_{\min}(\mathbf{x}_*) = 1.339956367. \quad (51)$$

This is the global optimum. However, the authors have not seen any studies that have found this solution in the literature. Slightly higher values have been found by cuckoo search and other methods [[Chickermane and Gea (1996), Gandomi et al. (2013)]]. The best solution found so far by [Gandomi et al. (2013)] is

$$\mathbf{x}_{\text{best}} = (6.0089, 5.3049, 4.5023, 3.5077, 2.1504), \quad (52)$$

and

$$f_{\text{best}} = 1.33999, \quad (53)$$

which is near this global optimum.

The above mathematical analysis can be very useful to guide future validation of new optimization methods when the above design benchmarks are used.

## 6 Conclusions

Pressure vessel design problem is a well-tested benchmark that has been used for validating optimization algorithms and their performance. We have provided a detailed mathematical analysis and obtained its global optimality. We have also used the method of Lagrange multipliers to double-check that the obtained optimum is indeed the global optimum for the pressure vessel design problem. By using the same methodology, we also analysed the design optimization of a cantilever beam.

However, it is worth pointing out that the method of Lagrange multipliers is only valid for optimization problems with equalities or when an inequality becomes tight. For general nonlinear optimization problems, we have to use the full Karush-Kuhn-Tucker (KKT) conditions to analyze their optimality [[Yang (2010)]], though such KKT can be extremely challenging to analyse in practice.

Even for design problems with only a few design variables, an analytical solution will provide greater insight into the problem and thus can act as better benchmarks for validating new optimization algorithms. Further work can focus on the analysis of other nonlinear design benchmarks.



## References

- [Akhtar et al. (2002)] Akhtar, S., Tai, K., Ray, T. (2002). A socio-behavioural simulation model for engineering design optimization, *Int Engineering Optimization*, **34**(4), pp. 341–354.
- [Annaratone (2007)] Annaratone, D. (2007) Pressure Vessel Design, Springer-Verlag Berlin Heidelberg.
- [Cagnina et al. (2008)] Cagnina L. C., Esquivel S. C., and Coello C. A., (2008). Solving engineering optimization problems with the simple constrained particle swarm optimizer, *Informatica*, **32**, 319–326.
- [Cai and Thierauf (1997)] Cai, J., Thierauf, G., (1997). Evolution strategies in engineering optimization, *Eng Optimization*, **29**(1), pp. 177–199.
- [Cao and Wu (1997)] Cao, Y. J., Wu, Q. H., (1997). Mechanical design optimization by mixed variable evolutionary programming. In: *Proceedings of the 1997 International Conference on Evolutionary Computation*, Indianapolis, pp. 443–446.
- [Cao and Wu (1999)] Cao, Y. J., Wu, Q. H., (1999). A mixed variable evolutionary programming for optimization of mechanical design, *Int J Eng Intel Syst Elect Eng Commun*, **7**(2), pp. 77–82.
- [Che and Cui (2011)] Che, Z. H., Cui, Z. H., (2011). Unbalanced supply chain design using the analytic network process and a hybrid heuristic-based algorithm with balance modulating mechanism, *Int. J. Bio-inspired Computation*, **3**(1), 56–66.
- [Chickermane and Gea (1996)] Chickermane, H., Gea, H. C., (1996). Structural optimization using a new local approximation method, *Int J Numer Method Eng*, **39**, pp.829–846.
- [Santos Ceolho (2010)] dos Santos Coelho, L., (2010). Gaussian quantum-behaved particle swarm optimization approaches for constrained engineering design problems, *Expert. Syst. Appl.*, **37**(2), pp. 1676–1683.
- [Coello (2000a)] Coello C. A. C., (2000a). Use of a self-adaptive penalty approach for engineering optimization problems. *Comput. Ind.*, **41**(2), pp. 113–127.
- [Coello (2000b)] Coello, C. A. C., (2000b). Constraint-handling using an evolutionary multiobjective optimization technique, *Civil Engrg Environ Syst*, **17**, pp. 319–346.
- [Coello (1999)] Coello C. A. C., (1999). Self-adaptive penalties for GA based optimization, *Proc. Congr. Evol. Comput.*, **1**, pp. 573–580.
- [Coello and Cortés (2004)] Coello, C. A. C., Cortés, N. C., (2004). Hybridizing a genetic algorithm with an artificial immune system for global optimization, *Engineering Optimization*, **36**(5), pp. 607–634.
- [Coello and Mezura Montes (2001)] Coello, C. A. C., Mezura Montes, E., (2001). Use of dominance-based tournament selection to handle constraints in genetic algorithms. In: *Intelligent Engineering Systems through Artificial Neural Networks* (ANNIE2001), **11**, ASME Press, St. Louis, pp. 177–182.
- [Cui et al. (2013)] Cui, Z. H., Fan S. J., Zeng J. C., Shi, Z. Z., (2013). Artificial plant optimisation algorithm with three-period photosynthesis, *Int. J. Bio-Inspired Computation*, **5**(2), 133–139.
- [Deb and Gene (1997)] Deb, K., Gene, A. S., (1997). A robust optimal design technique for mechanical component design. in: *Evolutionary algorithms in engineering applications*. Springer-Verlag, Berlin, pp. 497–514.
- [Fleury and Braibant (1986)] Fleury, C., Braibant, V., (1986). Structural optimization: a new dual method using mixed variables, *Int J Numer Meth Eng*, **23**, pp. 409–428.

- [Fu et al. (1991)] Fu, J., Fenton, R. G., Cleghorn, W. L., (1991). A mixed integer-discrete continuous programming method and its application to engineering design optimization, *Engineering Optimization*, **17**, pp. 263–280.
- [Gandomi et al. (2013)] Gandomi, A. H., Yang, X.-S., and Alavi, A. H., (2013). Cuckoo search algorithm: a metaheuristic approach to solve structural optimization problems, *Engineering with Computers*, **29**(1), pp. 17–35.
- [Gandomi et al. (2011)] Gandomi, A. H., Yang, X. S., Alavi, A. H., (2011). Mixed variable structural optimization using Firefly Algorithm, *Computers & Structures*, **89** (23-24), 2325–2336.
- [Gandomi and Yang (2011)] Gandomi, A. H. and Yang, X. S., (2011). Benchmark problems in structural optimization, in: *Computational Optimization, Methods and Algorithms*, Studies in Computational Intelligence (Eds. Koziel S. and Yang X. S.), Springer, Heidelberg, vol. 356, pp. 259–281.
- [Gandomi et al. (2012)] Gandomi, A. H., Yang, X. S., Talatahari, S., Deb, S., (2012). Coupled eagle strategy and differential evolution for unconstrained and constrained global optimization, *Computers & Mathematics with Applications*, **63**(1), 191–200.
- [Hadj-Alouane and Bean (1997)] Hadj-Alouane, A. B., Bean, J. C., (1997). A genetic algorithm for the multiple-choice integer program. *Oper Res*, **45**, pp. 92–101.
- [He et al. (2004)] He, S., Prempan, E., Wu, Q. H., (2004). An improved particle swarm optimizer for mechanical design optimization problems, *Engineering Optimization*, **36**(5), pp. 585–605.
- [He and Wang (2006)] He, Q., Wang, L., (2006). An effective co-evolutionary particle swarm optimization for engineering optimization problems, *Eng Appl Artif Intel*, **20**, pp.89–99.
- [Hu et al. (2003)] Hu, X., Eberhart, R. C., Shi, Y. (2003). Engineering optimization with particle swarm. In: *Proc. 2003 IEEE Swarm Intelligence Symposium*, pp. 53–57.
- [Huang et al. (2007)] Huang, F. Z., Wang, L., He, Q. (2007). An effective co-evolutionary differential evolution for constrained optimization, *Appl. Math. Comput.*, **186**, pp. 340–356.
- [Jamil and Yang (2013)] Jamil, M., and Yang, X. S., (2013). A literature survey of benchmark functions for global optimisation problems, *Int. J. of Mathematical Modelling and Numerical Optimisation*, **4**(2), 150–194.
- [Joines and Houck (1994)] Joines, J., Houck, C. (1994). On the use of non-stationary penalty functions to solve nonlinear constrained optimization problems with GAs. In: *Proceedings of the first IEEE Conference on Evolutionary Computation*, Orlando, Florida. D. Fogel (ed.). IEEE Press, pp 579–584
- [Kannan and Kramer (1994)] Kannan, B. K., Kramer, S. N., (1994). An augmented Lagrange multiplier based method for mixed integer discrete continuous optimization and its applications to mechanical design. *J Mech Des. Trans.*, **116**, pp. 318–320
- [Kaveh and Talatahari (2010)] Kaveh, A., Talatahari, S., (2010). An improved ant colony optimization for constrained engineering design problems, *Engineering Computations*, **27**(1), pp.155–182.
- [Lee and Geem (2005)] Lee K. S., Geem Z. W. (2005), A new meta-heuristic algorithm for continuous engineering optimization: harmony search theory and practice, *Comput. Methods Appl. Mech. Eng.*, **194**, pp. 3902–3933.
- [Li and Chou (1994)] Li, H.-L., Chou C.-T., (1994). A global approach for nonlinear mixed discrete programming in design optimization. *Engineering Optimization*, **22**, pp. 109–122.

- [Li and Chang (1998)] Li, H. L., Chang, C. T., (1998). An approximate approach of global optimization for polynomial programming problems, *Eur J. Oper Res*, **107**(3), pp.625–632.
- [Litinetskiand Abramzon (1998)] Litinetski, V. V., Abramzon, B. M., (1998). Multistart adaptive random search method for global constrained optimization in engineering applications, *Engineering Optimization*, **30**(2), pp. 125–154.
- [Michalewicz and Attia (1994)] Michalewicz, Z., Attia, N. (1994). Evolutionary optimization of constrained problems. Proceedings of the 3rd Annual Conference on Evolutionary Programming, World Scientific, pp. 98–108.
- [Montes et al. (2007)] Montes, E. M., Coello, C. A. C., Velázquez-Reyes, J, Muñoz-Dávila, L., (2007). Multiple trial vectors in differential evolution for engineering design, *Engineering Optimization*, **39**(5), pp. 567–589.
- [Parsopoulos and Vrahatis (2005)] Parsopoulos, K. E., Vrahatis, M. N.,. (2005). Unified particle swarm optimization for solving constrained engineering optimization problems. In: *Lecture Notes in Computer Science (LNCS)*, **3612**, pp. 582–591.
- [Rat and Liew (2003)] Ray, T., Liew, K., (2003). Society and civilization: An optimization algorithm based on the simulation of social behavior, *IEEE Trans Evol Comput*, **7**(4), pp.386–396.
- [Sandgren (1998)] Sandgren, E., (1988). Nonlinear integer and discrete programming in mechanical design, *Proceedings of the ASME Design Technology Conference*, Kissimine, FL, pp. 95–105.
- [Sandgren (1990)] Sandgren, E. (1990). Nonlinear integer and discrete programming in mechanical design optimization, *J Mech Design*, **112**(2), pp. 223–229.
- [Shih and Lai (1995)] Shih, C. J., Lai, T. K., (1995). Mixed-discrete fuzzy programming for nonlinear engineering optimization, *Engineering Optimization*, **23**(3), pp. 187–199.
- [Thanedar and Vanderplaats (1995)] Thanedar, P. B., Vanderplaats, G. N., (1995). Survey of discrete variable optimization for structural design, *Journal of Structural Engineering ASCE*, **121** (2), 301–306.
- [Tsai et al. (2002)] Tsai, J.-F., Li, H.-L., Hu, N.-Z., (2002). Global optimization for signomial discrete programming problems in engineering design, *Engineering Optmization*, **34**(6), pp. 613–622.
- [Wu and Chow (1995)] Wu, S. J., Chow, P. T., (1995). Genetic algorithms for nonlinear mixed discrete-integer optimization problems via meta-genetic parameter optimization, *Engineering Optimization*, **24**, pp. 137–159.
- [Yang (2010)] Yang X. S., (2010). *Engineering Optimisation: An Introduction with Metaheuristic Applications*, John Wiley and Sons.
- [Yang and Gandomi (2012)] Yang, X. S. and Gandomi, A. H., (2012). Bat algorithm: a novel approach for global engineering optimization, *Engineering Computations*, **29**(5), pp. 464–483.
- [Yang (2013)] Yang, X. S., (2013). Multiobjective firefly algorithm for continuous optimization, *Engineering with Computers*, **29**(2), 175–184.
- [Yang and Deb (2013)] Yang, X. S. and Deb, S., (2013). Multiobjective cuckoo search for design optimization, *Computers & Operations Research*, **40**(6), 1616–1624.
- [Yun (2005)] Yun, Y. S., (2005). Study on Adaptive Hybrid Genetic Algorithm and Its Applications to Engineering Design Problems, Waseda University, MSc Thesis.
- [Zhang and Wang (1993)] Zhang, C., Wang, H. P., (1993). Mixed-discrete nonlinear optimization with simulated annealing. *Engineering Optmization*, **17**(3), pp. 263280.